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# Congruences modulo 3 for two interesting partitions arising from two theta function identities

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**Abstract.** We find several interesting congruences modulo 3 for 5-core partitions and two color partitions.

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## Introduction

A partition of  $n$  is a non-increasing sequence of positive integers, called parts, whose sum is  $n$ . A partition of  $n$  is called a  $t$ -core of  $n$  if none of the hook numbers is a multiple of  $t$ . If  $a_t(n)$  denotes the number of  $t$ -cores of  $n$ , then the generating function for  $a_t(n)$  is [5]

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t; q^t)_{\infty}^t}{(q; q)_{\infty}}, \quad (1)$$

where, here and throughout the sequel, for any complex number  $a$  and  $|q| < 1$ ,

$$(a; q)_{\infty} := \prod_{n=1}^{\infty} (1 - q^n).$$

The following exact formula for  $a_5(n)$  in terms of the prime factorization of  $n + 1$  can be found in [5, Theorem 4]. This theorem follows from an identity of Ramanujan recorded in his famous manuscript on the partition function and tau-function now published with his lost notebook [6, p. 139].

**Theorem 1.** *Let  $n + 1 = 5^c p_1^{a_1} \dots p_s^{a_s} q_1^{b_1} \dots q_t^{b_t}$  be the prime factorization of  $n + 1$  into primes  $p_i \equiv 1, 4 \pmod{5}$ , and  $q_j \equiv 2, 3 \pmod{5}$ . Then*

$$a_5(n) = 5^c \prod_{i=1}^s \frac{p_i^{a_i+1} - 1}{p_i - 1} \prod_{j=1}^t \frac{q_j^{b_j+1} - 1}{q_j - 1}.$$

Many arithmetical identities and congruences easily follow from the above theorem. For example, for any positive integer  $n$  and non-negative integer  $r$ , we have

$$a_5(5^\alpha n - 1) = 5^\alpha a_5(n - 1) \equiv 0 \pmod{5^\alpha}.$$

Similarly, for a prime  $p \equiv 2, 3 \pmod{5}$  and any non-negative integers  $n$  and  $r$ , we can easily deduce that

$$a_5(p^\alpha(pn + 1) - 1) = \frac{p^{\alpha+1} + (-1)^\alpha}{p + 1} a_5(pn) \equiv 0 \pmod{\frac{p^{\alpha+1} + (-1)^\alpha}{p + 1}}.$$

Next, let  $p_k(n)$  denote the number of 2-color partitions of  $n$  where one of the colors appears only in parts that are multiples of  $k$ . Then the generating function for  $p_k(n)$  is given by

$$\sum_{n=0}^{\infty} p_k(n) q^n = \frac{1}{(q; q)_\infty (q^k; q^k)_\infty}.$$

Recently, the following result was proved by Baruah, Ahmed and Dastidar [1].

**Theorem 2.** *If  $k \in \{0, 1, 2, 3, 4, 5, 10, 15, 20\}$ , then for any non-negative integer  $n$ ,*

$$p_k(25n + \ell) \equiv 0 \pmod{5},$$

where  $k + \ell = 24$ .

In this paper, we find some interesting congruences modulo 3 for 5-core partitions and  $p_5(n)$  by employing Ramanujan's theta functions and their dissections. Since our proofs mainly rely on various properties of Ramanujan's theta functions and dissections of certain  $q$ -products, we end this section by defining a  $t$ -dissection and Ramanujan's general theta function and some of its special cases. If  $P(q)$  denotes a power series in  $q$ , then a  $t$ -dissection of  $P(q)$  is given by

$$P(q) = \sum_{k=0}^{t-1} q^k P_k(q^t),$$

where  $P_k$  are power series in  $q^t$ . In the remainder of this section, we introduce

Ramanujan's theta functions and some of their elementary properties, which will be used in our subsequent sections. For  $|ab| < 1$ , Ramanujan's general theta-function  $f(a, b)$  is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

In this notation, Jacobi's famous triple product identity [3, p. 35, Entry 19] takes the form

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}. \quad (2)$$

Three important special cases of  $f(a, b)$  are

$$\varphi(q) := f(q, q) = \frac{(q^2; q^2)_{\infty} (-q; q^2)_{\infty}}{(q; q^2)_{\infty} (-q^2; q^2)_{\infty}} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2}, \quad (3)$$

$$\psi(q) := f(q, q^3) = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \quad (4)$$

and

$$f(-q) := f(-q, -q^2) = (q; q)_{\infty}, \quad (5)$$

where the product representations in (4) and (5) arise from (2) and the last equality in (5) is Euler's famous pentagonal number theorem. After Ramanujan, we also define

$$\chi(q) := (-q; q^2)_{\infty} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty} (q^4; q^4)_{\infty}} \quad (6)$$

and hence,

$$\chi(-q) := (q; q^2)_{\infty} = \frac{(q; q)_{\infty}}{(q^2; q^2)_{\infty}} \quad (7)$$

Furthermore, the  $q$ -product representations of  $\varphi(-q)$  and  $\psi(-q)$  can be written in the forms

$$\varphi(-q) = \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}}, \quad \psi(-q) = \frac{(q; q)_{\infty} (q^4; q^4)_{\infty}}{(q^2; q^2)_{\infty}} \quad (8)$$

**Lemma 1.** [4, Theorem 2.2] *For any prime  $p \geq 5$ ,*

$$(q; q)_\infty = \sum_{\substack{k = -\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) \\ + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2}; q^{p^2})_\infty, \quad (9)$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}; \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Furthermore, for  $-\frac{(p-1)}{2} \leq k \leq \frac{(p-1)}{2}$  and  $k \neq \frac{(\pm p-1)}{6}$ ,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

## Some congruences modulo 3 for 5-core partitions

We first introduce an important lemma which will be used later.

**Lemma 2.** *We have*

$$(q^2; q^2)_\infty^2 (q^{10}; q^{10})_\infty^2 \equiv q^2 (q^6; q^6)_\infty^2 (q^{30}; q^{30})_\infty^2 + (q^6; q^6)_\infty^4 + 2q^4 (q^{30}; q^{30})_\infty^4 \pmod{3}. \quad (10)$$

*Proof.* We note that [3, p. 258],

$$\varphi^2(q) - \varphi^2(q^5) = 4q\chi(q)f(-q^5)f(-q^{20}) \quad (11)$$

and

$$5\varphi^2(q^5) - \varphi^2(q) = 4\chi(q)\chi(-q^5)\psi^2(-q). \quad (12)$$

Multiplying (11) and (12)

$$6\varphi^2(q)\varphi^2(q^5) - \varphi^4(q) - 5\varphi^4(q^5) = 16q\chi^2(q)\chi(q^5)f(-q^5)f(-q^{20})\psi(-q). \quad (13)$$

Employing (5), (6), (7) and (8) in the last equation, we obtain

$$6\varphi^2(q)\varphi^2(q^5) - \varphi^4(q) - 5\varphi^4(q^5) = 16q(q^2; q^2)_\infty^2 (q^{10}; q^{10})_\infty^2. \quad (14)$$

Taking congruences modulo 3, we obtain

$$q(q^2; q^2)_\infty^2 (q^{10}; q^{10})_\infty^2 \equiv -\varphi(q)\varphi(q^3) + \varphi(q^5)\varphi(q^{15}) \pmod{3}. \quad (15)$$

Recalling [3, p.49, Corollary (i)], we have

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}) \quad (16)$$

Replacing  $q$  by  $q^5$  in the above equation, we have,

$$\varphi(q^5) = \varphi(q^{45}) + 2q^5f(q^{15}, q^{75}) \quad (17)$$

Employing the above two equation in (15), we find that

$$\begin{aligned} q(q^2; q^2)_\infty^2 (q^{10}; q^{10})_\infty^2 &\equiv -\varphi(q^3)\varphi(q^9) + q\varphi(q^3)f(q^3, q^{15}) + \varphi(q^{15})\varphi(q^{45}) \\ &\quad + 2q^5\varphi(q^{15})f(q^{15}, q^{75}) \pmod{3}. \end{aligned} \quad (18)$$

Note that

$$\varphi(q)f(q, q^5) = \frac{(q^2; q^2)_\infty^7 (q^3; q^3)_\infty (q^{12}; q^{12})_\infty}{(q; q)_\infty^3 (q^4; q^4)_\infty^3 (q^6; q^6)_\infty} \quad (19)$$

Since,  $(q; q)_\infty^3 \equiv (q^3; q^3)_\infty$ , so (19) can be written as

$$\varphi(q)f(q, q^5) \equiv (q^2; q^2)_\infty^4 \pmod{3}. \quad (20)$$

Employing (15) and (20) in (18), we obtain

$$\begin{aligned} q(q^2; q^2)_\infty^2 (q^{10}; q^{10})_\infty^2 &\equiv \\ &\quad q^3(q^6; q^6)_\infty^2 (q^{30}; q^{30})_\infty^2 + q(q^6; q^6)_\infty^4 + 2q^5(q^{30}; q^{30})_\infty^4 \pmod{3}. \end{aligned} \quad (21)$$

From above congruence we can easily obtain (10).  $\square$

**Theorem 3.** *We have,*

$$\sum_{n=0}^{\infty} a_5(3n)q^n \equiv (q^3; q^3)_\infty (q^5; q^5)_\infty \pmod{3}, \quad (22)$$

$$\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv (q; q)_\infty (q^{15}; q^{15})_\infty \pmod{3}, \quad (23)$$

and

$$\sum_{n=0}^{\infty} a_5(3n+2)q^n \equiv 2 \sum_{n=0}^{\infty} a_5(n)q^n \pmod{3}. \quad (24)$$

*Proof.* Putting  $t = 5$  in (3.7) and replacing  $q$  by  $q^2$ , we have

$$\sum_{n=0}^{\infty} a_5(n) q^{2n} = \frac{(q^{10}; q^{10})_{\infty}^5}{(q^2; q^2)_{\infty}}. \quad (25)$$

Taking congruences modulo 3,

$$\sum_{n=0}^{\infty} a_5(n) q^{2n} \equiv \frac{(q^{30}; q^{30})_{\infty} (q^2; q^2)_{\infty}^2 (q^{10}; q^{10})_{\infty}^2}{(q^6; q^6)_{\infty}} \pmod{3}. \quad (26)$$

With the help of (10), (26) can be rewritten as

$$\sum_{n=0}^{\infty} a_5(n) q^n \equiv \frac{(q^{30}; q^{30})_{\infty}}{(q^6; q^6)_{\infty}} (q^2 (q^6; q^6)_{\infty}^2 (q^{30}; q^{30})_{\infty}^2 \quad (27)$$

$$+ (q^6; q^6)_{\infty}^4 + 2q^4 (q^{30}; q^{30})_{\infty}^4) \pmod{3} \\ \equiv q^2 (q^6; q^6)_{\infty} (q^{30}; q^{30})_{\infty}^3 + (q^6; q^6)_{\infty}^3 (q^{30}; q^{30})_{\infty} \quad (28)$$

$$+ 2q^4 \frac{(q^{30}; q^{30})_{\infty}^5}{(q^6; q^6)_{\infty}} \pmod{3} \\ \equiv q^2 (q^6; q^6)_{\infty} (q^{90}; q^{90})_{\infty} + (q^{18}; q^{18})_{\infty} (q^{30}; q^{30})_{\infty} \quad (29)$$

$$+ 2q^4 \sum_{n=0}^{\infty} a_5(n) q^{6n} \pmod{3}.$$

Comparing the terms involving  $q^{6n}$ ,  $q^{6n+2}$  and  $q^{6n+4}$  respectively from both sides of the above congruence, we can easily obtain (22)–(24).  $\square$

Applying the mathematical induction in (24), we can easily obtain the following

**Corollary 1.** *For any nonnegative integers  $k$  and  $n$ , we have*

$$a_5(3^k n + 3^k - 1) \equiv 2^k a_5(n) \pmod{3}.$$

**Theorem 4.** *We have,*

$$a_5(15n + 6) \equiv 0 \pmod{3}, \quad (30)$$

$$a_5(15n + 12) \equiv 0 \pmod{3}, \quad (31)$$

and

$$a_5(15n + 9) \equiv 2a_5(3n + 1) \pmod{3}. \quad (32)$$

*Proof.* From [3, p. 270, Entry 12(v)], we recall that

$$(q; q)_\infty = (q^{25}; q^{25})_\infty \left( \frac{A(q^5)}{B(q^5)} - q - q^2 \frac{B(q^5)}{A(q^5)} \right) \quad (33)$$

where

$$A(q) = \frac{f(-q^{10}, -q^{15})}{f(-q^5, -q^{20})} \text{ and } B(q) = \frac{f(-q^5, -q^{20})}{f(-q^{10}, -q^{15})}.$$

Replacing  $q$  by  $q^3$  in (33) and then employing in (22), we obtain

$$\sum_{n=0}^{\infty} a_5(3n)q^n \equiv (q^5; q^5)_\infty (q^{75}; q^{75})_\infty \left( \frac{A(q^{15})}{B(q^{15})} - q^3 - q^6 \frac{B(q^{15})}{A(q^{15})} \right) \pmod{3} \quad (34)$$

Extracting the terms involving  $q^{5n+3}$  from both sides of the congruence, we obtain,

$$\sum_{n=0}^{\infty} a_5(3(5n+3))q^n \equiv 2(q; q)_\infty (q^{15}; q^{15})_\infty \pmod{3} \quad (35)$$

Employing (23) in (35), we can easily obtain (32).

We have seen that in the right hand side of the congruence (34), there is no terms involving  $q^{5n+2}$  and  $q^{5n+4}$  and hence we can easily obtain (30) and (31) respectively.  $\square$

**Theorem 5.** *We have,*

$$a_5(15n+4) \equiv 2a_5(3n) \pmod{3}, \quad (36)$$

$$a_5(15n+10) \equiv 0 \pmod{3}, \quad (37)$$

and

$$a_5(15n+13) \equiv 0 \pmod{3}. \quad (38)$$

*Proof.* Employing (33) in (23), we obtain

$$\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv (q^{15}; q^{15})_\infty (q^{25}; q^{25})_\infty \left( \frac{A(q^5)}{B(q^5)} - q - q^2 \frac{B(q^5)}{A(q^5)} \right) \pmod{3} \quad (39)$$

Extracting the terms involving  $q^{5n+1}$  from both sides of the congruence, we obtain,

$$\sum_{n=0}^{\infty} a_5(3(5n+1)+1)q^n \equiv 2(q^3; q^3)_{\infty}(q^5; q^5)_{\infty} \pmod{3} \quad (40)$$

Employing (22) in (40), we can easily obtain (36).

It is clear that in the right hand side of the congruence (39), there is no terms involving  $q^{5n+3}$  and  $q^{5n+4}$  and hence we can easily obtain (37) and (38)

□ QED

**Theorem 6.** For any prime  $p \geq 5$  with  $\left(\frac{-15}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,

$$a_5\left(3 \cdot p^{2k}n + p^{2k} - 1\right) \equiv a_5(3n) \pmod{3}. \quad (41)$$

*Proof.* With the help of (9), (22) can be rewritten as

$$\begin{aligned} \sum_{n=0}^{\infty} a_5(3n)q^n &\equiv \left[ \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{3 \cdot \frac{3k^2+k}{2}} f\left(-q^{3 \cdot \frac{3p^2+(6k+1)p}{2}}, -q^{3 \cdot \frac{3p^2-(6k+1)p}{2}}\right) + \right. \\ &\quad \left. (-1)^{\frac{\pm p-1}{6}} q^{3 \cdot \frac{p^2-1}{24}} f(-q^{3p^2}) \right] \\ &\quad \times \left[ \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{5 \cdot \frac{3k^2+k}{2}} f\left(-q^{5 \cdot \frac{3p^2+(6k+1)p}{2}}, -q^{5 \cdot \frac{3p^2-(6k+1)p}{2}}\right) \right. \\ &\quad \left. + (-1)^{\frac{\pm p-1}{6}} q^{5 \cdot \frac{p^2-1}{24}} f(-q^{5p^2}) \right] \pmod{3}. \end{aligned} \quad (42)$$

Now we consider the congruence

$$3 \cdot \frac{(3k^2+k)}{2} + 5 \cdot \frac{(3m^2+m)}{2} \equiv 8 \cdot \frac{(p^2-1)}{24} \pmod{p}, \quad (43)$$

where  $-(p-1)/2 \leq k, m \leq (p-1)/2$ , with  $\left(\frac{-15}{p}\right) = -1$ . Since the above congruence is equivalent to

$$(18k+3)^2 + 15 \cdot (6m+1)^2 \equiv 0 \pmod{p},$$



and  $\left(\frac{-15}{p}\right) = -1$ , there is only one solution  $k = m = \pm p - 1/6$  for (43).

That is, there are no other  $k$  and  $m$  such that  $3 \frac{(3k^2 + k)}{2} + 5 \cdot \frac{(3m^2 + m)}{2}$  and  $8 \cdot \frac{(p^2 - 1)}{24}$  are in the same residue class modulo  $p$ . Therefore, equating the terms involving  $q^{pn+8 \cdot \frac{p^2-1}{24}}$  from both sides of (42), we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} a_5 \left( 3 \left( pn + 8 \cdot \frac{p^2 - 1}{24} \right) \right) q^n &= \sum_{n=0}^{\infty} a_5 (3pn + p^2 - 1) q^n \\ &\equiv (q^{3p}; q^{3p})_{\infty} (q^{5p}; q^{5p})_{\infty} \pmod{3}. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} a_5 (3p^2n + p^2 - 1) q^n \equiv (q^3; q^3)_{\infty} (q^5; q^5)_{\infty} \pmod{3}. \quad (44)$$

From (44) and (22), we arrive at

$$a_5 (3p^2n + p^2 - 1) \equiv a_5(3n) \pmod{3}.$$

Now (41) can be established easily by mathematical induction.  $\square$  *QED*

**Corollary 2.** *For any prime  $p \geq 5$  with  $\left(\frac{-15}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,*

$$a_5 \left( 3 \cdot p^{2k+2}n + (3i + p)p^{2k+1} - 1 \right) \equiv 0 \pmod{3},$$

where  $i = 1, 2, \dots, p - 1$ .

*Proof.* As in the proof of the previous theorem, it can also be shown that

$$\sum_{n=0}^{\infty} a_5 \left( 3 \cdot p^{2k} \left( pn + 8 \frac{p^2 - 1}{24} \right) + p^{2k} - 1 \right) q^n \equiv (q^{3p}; q^{3p})_{\infty} (q^{5p}; q^{5p})_{\infty} \pmod{3},$$

that is,

$$\sum_{n=0}^{\infty} a_5 \left( 3 \cdot p^{2k+1}n + p^{2k+2} - 1 \right) q^n \equiv (q^{3p}; q^{3p})_{\infty} (q^{5p}; q^{5p})_{\infty} \pmod{3}.$$

Since there are no terms on the right side of the above congruence in which the powers of  $q$  are congruent to  $1, 2, \dots, p - 1$  modulo  $p$ , it follows, for  $i = 1, 2, \dots, p - 1$ , that

$$a_5 \left( 3 \cdot p^{2k+1}(pn + i) + p^{2k+2} - 1 \right) \equiv 0 \pmod{3},$$

which is clearly equivalent to the proffered congruence.  $\square$  *QED*

**Corollary 3.** *For any prime  $p \geq 5$  with  $\left(\frac{-15}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,*

$$a_5 \left( 15 \cdot p^{2k}n + (3r + 1)p^{2k} - 1 \right) \equiv 0 \pmod{3},$$

where  $r = 2, 4$ .

*Proof.* It can also be shown that

$$\sum_{n=0}^{\infty} a_5 \left( 3 \cdot p^{2k}n + p^{2k} - 1 \right) q^n \equiv (q^3; q^3)_{\infty} (q^5; q^5)_{\infty} \pmod{3}.$$

In (33), replacing  $q$  by  $q^3$ , we can see that  $(q^3; q^3)_{\infty} (q^5; q^5)_{\infty}$  has no terms  $q^{5n+r}$  where  $r = 2, 4$ , it follows that

$$a_5 \left( 3 \cdot p^{2k}(5n + r) + p^{2k} - 1 \right) \equiv 0 \pmod{3},$$

which is clearly equivalent to the proffered congruence.  $\square$

**Theorem 7.** *For any prime  $p \geq 5$  with  $\left(\frac{-15}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,*

$$a_5 \left( 3 \cdot p^{2k}n + 2 \cdot p^{2k} - 1 \right) \equiv a_5(3n + 1) \pmod{3}. \quad (45)$$

*Proof.* With the help of (9), (23) can be rewritten as

$$\begin{aligned} \sum_{n=0}^{\infty} a_5(3n + 1)q^n &\equiv \left[ \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + \right. \\ &\quad \left. (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}) \right] \\ &\quad \times \left[ \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{15 \cdot \frac{3k^2+k}{2}} f\left(-q^{15 \cdot \frac{3p^2+(6k+1)p}{2}}, -q^{15 \cdot \frac{3p^2-(6k+1)p}{2}}\right) \right. \\ &\quad \left. + (-1)^{\frac{\pm p-1}{6}} q^{15 \cdot \frac{p^2-1}{24}} f(-q^{15p^2}) \right] \pmod{3}. \end{aligned} \quad (46)$$

Consider the congruence

$$\frac{(3k^2 + k)}{2} + 15 \cdot \frac{(3m^2 + m)}{2} \equiv 16 \cdot \frac{(p^2 - 1)}{24} \pmod{p}, \quad (47)$$

where  $-(p-1)/2 \leq k$ ,  $m \leq (p-1)/2$ , with  $\left(\frac{-15}{p}\right) = -1$ . Since the above congruence is equivalent to

$$(6k+1)^2 + 15 \cdot (6m+1)^2 \equiv 0 \pmod{p},$$

and  $\left(\frac{-15}{p}\right) = -1$ , there is only one solution  $k = m = \pm p - 1/6$  for (47).

That is, there are no other  $k$  and  $m$  such that  $\frac{(3k^2+k)}{2} + 15 \cdot \frac{(3m^2+m)}{2}$  and  $16 \cdot \frac{(p^2-1)}{24}$  are in the same residue class modulo  $p$ . Therefore, equating the terms involving  $q^{pn+16 \cdot \frac{p^2-1}{24}}$  from both sides of (46), we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} a_5 \left( 3 \left( pn + 16 \cdot \frac{p^2-1}{24} \right) + 1 \right) q^n &= \sum_{n=0}^{\infty} a_5 (3pn + 2p^2 - 1) q^n \\ &\equiv (q^p; q^p)_{\infty} (q^{15p}; q^{15p})_{\infty} \pmod{3}. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} a_5 (3p^2n + 2p^2 - 1) q^n \equiv (q; q)_{\infty} (q^{15}; q^{15})_{\infty} \pmod{3}. \quad (48)$$

From (48) and (23), we arrive at

$$a_5 (3p^2n + 2p^2 - 1) \equiv a_5(3n + 1) \pmod{3}.$$

Now (45) can be established easily by mathematical induction.  $\square$

The following results follow in a similar fashion. So we omit the proof.

**Corollary 4.** *For any prime  $p \geq 5$  with  $\left(\frac{-15}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,*

$$a_5 \left( 3 \cdot p^{2k+2} + (3i + 2p)p^{2k+1} - 1 \right) \equiv 0 \pmod{3},$$

where  $i = 1, 2, \dots, p-1$ .

**Corollary 5.** *For any prime  $p \geq 5$  with  $\left(\frac{-15}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,*

$$a_5 \left( 15 \cdot p^{2k} + (3r + 2)p^{2k} - 1 \right) \equiv 0 \pmod{3},$$

where  $r = 3, 4$ .

## Some congruences modulo 3 for two color partitions $p_5(n)$

**Theorem 8.** *We have,*

$$\sum_{n=0}^{\infty} p_5(3n)q^n \equiv \frac{(q^3; q^3)_{\infty}}{(q^5; q^5)_{\infty}} \pmod{3}, \quad (49)$$

$$\sum_{n=0}^{\infty} p_5(3n+1)q^n \equiv (q; q)_{\infty} (q^5; q^5)_{\infty} \pmod{3}, \quad (50)$$

and

$$\sum_{n=0}^{\infty} p_5(3n+2)q^n \equiv 2 \frac{(q^{15}; q^{15})_{\infty}}{(q; q)_{\infty}} \pmod{3}. \quad (51)$$

where

$$\sum_{n=0}^{\infty} p_5(n)q^n := \frac{1}{(q; q)_{\infty} (q^5; q^5)_{\infty}}.$$

*Proof.* We have

$$\sum_{n=0}^{\infty} p_5(n)q^n = \frac{1}{(q; q)_{\infty} (q^5; q^5)_{\infty}} = \frac{(q^5; q^5)_{\infty}^5}{(q; q)_{\infty} (q^5; q^5)_{\infty}^6}.$$

Taking congruences modulo 3 in the above equation, we obtain,

$$\begin{aligned} \sum_{n=0}^{\infty} p_5(n)q^n &\equiv \frac{(q^5; q^5)_{\infty}^5}{(q^{15}; q^{15})_{\infty}^2 (q; q)_{\infty}} \pmod{3} \\ &\equiv \frac{1}{(q^{15}; q^{15})_{\infty}^2} \sum_{n=0}^{\infty} a_5(n)q^n \end{aligned} \quad (52)$$

Comparing the terms involving  $q^{3n}$ ,  $q^{3n+1}$  and  $q^{3n+2}$  respectively from the both sides of the above congruence, we have

$$\sum_{n=0}^{\infty} p_5(3n)q^n \equiv \frac{1}{(q^5; q^5)_{\infty}^2} \sum_{n=0}^{\infty} a_5(3n)q^n \pmod{3}, \quad (53)$$

$$\sum_{n=0}^{\infty} p_5(3n+1)q^n \equiv \frac{1}{(q^5; q^5)_{\infty}^2} \sum_{n=0}^{\infty} a_5(3n+1)q^n \pmod{3}, \quad (54)$$

and

$$\sum_{n=0}^{\infty} p_5(3n+2)q^n \equiv 2 \frac{1}{(q^5; q^5)_{\infty}^2} \sum_{n=0}^{\infty} a_5(3n+2)q^n \pmod{3}. \quad (55)$$

Employing (3) in the above congruences, we can easily obtain the (8).  $\square$

**Theorem 9.** *We have,*

$$p_5(15n+9) \equiv p_5(3n+2) \pmod{3}, \quad (56)$$

$$p_5(15n+6) \equiv 0 \pmod{3}, \quad (57)$$

and

$$p_5(15n+12)q^n \equiv 0 \pmod{3}. \quad (58)$$

*Proof.* Replacing  $q$  by  $q^3$  in (33) and then employing in (49), we obtain

$$\sum_{n=0}^{\infty} p_5(3n)q^n \equiv \frac{(q^{75}; q^{75})_{\infty}}{(q^5; q^5)_{\infty}} \left( \frac{A(q^{15})}{B(q^{15})} - q^3 - q^6 \frac{B(q^{15})}{A(q^{15})} \right) \pmod{3} \quad (59)$$

Comparing the terms involving  $q^{5n+3}$  from both sides of the above congruence we can easily arrive at (56). There is no terms involving  $q^{5n+2}$ ,  $q^{5n+4}$  in the right hand side of the congruence (59). So, we can easily obtain (57) and (58).  $\square$

**Theorem 10.** *For any  $k \geq 1$ , we have*

$$\sum_{n=0}^{\infty} p_5 \left( 3^{2k-1}n + \frac{3^{2k-1}+1}{4} \right) q^n \equiv 2^{k+1}(q; q)_{\infty}(q^5; q^5)_{\infty} \pmod{3}. \quad (60)$$

*Proof.* We prove the result by mathematical induction. For  $k = 1$ , we obtain

$$\sum_{n=0}^{\infty} p_5(3n+1)q^n \equiv (q; q)_{\infty}(q^5; q^5)_{\infty} \pmod{3}, \quad (61)$$

which is the (50).

Let (60) be true for some positive integer  $k$ . Therefore we can write (60) as

$$\sum_{n=0}^{\infty} p_5 \left( 3^{2k-1}n + \frac{3^{2k-1}+1}{4} \right) q^n \quad (62)$$

$$\equiv 2^{k+1}(q^3; q^3)_{\infty}(q^{15}; q^{15})_{\infty} \times \frac{1}{(q; q)_{\infty}^2 (q^5; q^5)_{\infty}^2} \pmod{3} \quad (63)$$

$$\equiv 2^{k+1}(q^3; q^3)_{\infty}(q^{15}; q^{15})_{\infty} \times \left( \sum_{n=0}^{\infty} p_5(n)q^n \right)^2 \pmod{3}.$$

Extracting the terms involving  $q^{3n+2}$  from both sides of the above congruence, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} p_5 \left( 3^{2k-1}(3n+2) + \frac{3^{2k-1}+1}{4} \right) q^n \equiv \\ 2^{k+1}(q; q)_{\infty}(q^5; q^5)_{\infty} \times \\ \left[ \left( \sum_{n=0}^{\infty} p_5(3n+1)q^n \right)^2 + 2 \sum_{n=0}^{\infty} p_5(3n)q^n \sum_{n=0}^{\infty} p_5(3n+2)q^n \right] \pmod{3}. \end{aligned} \quad (64)$$

Employing (49), (50) and (51) in (64),

$$\begin{aligned} \sum_{n=0}^{\infty} p_5 \left( 3^{2k-1}(3n+2) + \frac{3^{2k-1}+1}{4} \right) q^n \\ \equiv 2^{k+1}(q; q)_{\infty}(q^5; q^5)_{\infty} \left[ (q; q)_{\infty}^2 (q^5; q^5)_{\infty}^2 + 4 \frac{(q^3; q^3)_{\infty}(q^{15}; q^{15})_{\infty}}{(q^5; q^5)_{\infty}(q; q)_{\infty}} \right] \pmod{3} \\ \equiv 2^{k+2}(q; q)_{\infty}(q^5; q^5)_{\infty} \pmod{3}. \end{aligned} \quad (65)$$

Hence the result is true for all  $k \geq 1$ .  $\square$  *QED*

Applying (33) we can see easily that  $(q; q)_{\infty}(q^5; q^5)_{\infty}$  has no terms containing  $q^{5n+r}$  for  $r = 3, 4$ . From the last result we can conclude the following

**Theorem 11.** *For any  $k \geq 1$ , we have*

$$\sum_{n=0}^{\infty} p_5 \left( 3^{2k-1}n + \frac{(4r+1)3^{2k-1}+1}{4} \right) q^n \equiv 0 \pmod{3}, \quad (66)$$

where,  $r = 3, 4$ .

**Theorem 12.** *For any prime  $p \geq 5$  with  $\left(\frac{-5}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,*

$$p_5 \left( 3 \cdot p^{2k}n + \frac{3p^{2k}+1}{4} \right) \equiv p_5(3n+1) \pmod{3}. \quad (67)$$

*Proof.* With the help of (9), (50) can be rewritten as

$$\begin{aligned}
 \sum_{n=0}^{\infty} p_5(3n+1)q^n &\equiv \left[ \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + \right. \\
 &\quad \left. (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}) \right] \\
 &\quad \times \left[ \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{5 \cdot \frac{3k^2+k}{2}} f\left(-q^{5 \cdot \frac{3p^2+(6k+1)p}{2}}, -q^{5 \cdot \frac{3p^2-(6k+1)p}{2}}\right) \right. \\
 &\quad \left. + (-1)^{\frac{\pm p-1}{6}} q^{5 \cdot \frac{p^2-1}{24}} f(-q^{5p^2}) \right] \pmod{3}. \tag{68}
 \end{aligned}$$

Consider the following congruence

$$\frac{(3k^2+k)}{2} + 5 \cdot \frac{(3m^2+m)}{2} \equiv 6 \cdot \frac{(p^2-1)}{24} \pmod{p}, \tag{69}$$

where  $-(p-1)/2 \leq k, m \leq (p-1)/2$ , with  $\left(\frac{-5}{p}\right) = -1$ . Since the above congruence is equivalent to

$$(6k+1)^2 + 5 \cdot (6m+1)^2 \equiv 0 \pmod{p},$$

and  $\left(\frac{-5}{p}\right) = -1$ , there is only one solution  $k = m = \pm p-1/6$  for (47). That is,

there are no other  $k$  and  $m$  such that  $\frac{(3k^2+k)}{2} + 5 \cdot \frac{(3m^2+m)}{2}$  and  $6 \cdot \frac{(p^2-1)}{24}$  are in the same residue class modulo  $p$ . Therefore, equating the terms involving  $q^{pn + \frac{p^2-1}{4}}$  from both sides of (68), we deduce that

$$\begin{aligned}
 \sum_{n=0}^{\infty} p_5 \left( 3 \left( pn + \frac{p^2-1}{4} \right) + 1 \right) q^n &= \sum_{n=0}^{\infty} p_5 \left( 3pn + \frac{3p^2+1}{4} \right) q^n \\
 &\equiv (q^p; q^p)_{\infty} (q^{5p}; q^{5p})_{\infty} \pmod{3}.
 \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} p_5 \left( 3p^2n + \frac{3p^2+1}{4} \right) q^n \equiv (q; q)_{\infty} (q^5; q^5)_{\infty} \pmod{3}. \tag{70}$$

From (70) and (50), we arrive at

$$p_5 \left( 3p^2n + \frac{3p^2+1}{4} \right) \equiv p_5(3n+1) \pmod{3}.$$

Now (67) can be established easily by mathematical induction.  $\square$

The following results follow in a similar fashion. So we omit the proof.

**Corollary 6.** *For any prime  $p \geq 5$  with  $\left(\frac{-5}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,*

$$p_5 \left( 3 \cdot p^{2k+2} + \frac{(12i + 3p)p^{2k+1} + 1}{4} \right) \equiv 0 \pmod{3},$$

where  $i = 1, 2, \dots, p-1$ .

**Corollary 7.** *For any prime  $p \geq 5$  with  $\left(\frac{-5}{p}\right) = -1$  and for any non-negative integers  $k$  and  $n$ ,*

$$p_5 \left( 15 \cdot p^{2k} + \frac{(12r + 3)p^{2k} + 1}{4} \right) \equiv 0 \pmod{3},$$

where  $r = 3, 4$ .

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